

Phases of $N = 1$ supersymmetric gauge theories in four dimensions

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We exhibit $N = 1$ supersymmetric field theories in confining, Coulomb and Higgs phases. The superpotential and the gauge kinetic terms are holomorphic and can be determined exactly in the various phases. The Coulomb phase generically has points with massless monopoles. When they condense, the theory undergoes a phase transition to a confining phase. When there are points in the Coulomb phase with massless electric charges, their condensation leads to a transition to a Higgs phase. When the Higgs and confinement phases are distinct, we expect to find massless interacting gluons at the transition point between them.

1. Introduction

Gauge theories in four dimensions can manifest themselves in three known phases: Coulomb, Higgs, and confinement. As the parameters g_I of the theory are varied, a phase transition between them can take place. In supersymmetric theories there is a new phenomenon. When supersymmetry is unbroken the theory can have several inequivalent ground states for fixed values of the parameters. These different ground states can be in different phases. In many cases they form a continuous manifold – a moduli space of vacua [1] – parametrized by massless fields (moduli), Φ^r . There can then also be phase transitions in the space of vacua. Therefore, we will be interested in the total space whose base is the parameter space and the moduli space is fibered over it. (Note that for different values of the parameters the fiber can have different topology and even different dimension.)

A crucial element in our analysis is holomorphy. The superpotential W and the coefficient τ of the gauge kinetic terms are holomorphic in the moduli fields Φ^r . They must also be holomorphic in all the coupling constants g_I [2]. This holomorphy constrains the kind of transitions which are possible. First, there cannot be any first order transitions between different phases. This is because the energy of every ground state is zero when supersymmetry is unbroken and, hence, there are no level crossings of vacua. Furthermore, because of holomorphy, phase boundaries are always of real codimension two or larger. The known examples are of two classes:

1. Most of the moduli space is in one phase and it has a small subspace of complex codimension larger or equal to one of another phase [1].
2. The moduli space has separate branches in different phases which touch each other at transition points [3,4].

Below we will see more examples in both classes.

Most of the analysis of supersymmetric gauge theories has been devoted to theories with matter fields in the fundamental representation [5-10,1], where there is no distinction between the confining and the Higgs phase [11]. These theories did not have a manifold of ground states in the Coulomb phase. Other theories (like those of [3,4] and the ones below) do have a moduli space of vacua in the Coulomb phase. Trying to apply the techniques of [10] to these theories, one finds superpotentials which seem to miss the Coulomb phase. Below we will interpret these superpotentials as giving accurate descriptions of the confining phases of these theories.

One of the main points of this paper is an extension of some of the work of [3,4] from $N = 2$ to $N = 1$ theories. In the second section we discuss the Coulomb phase of $N = 1$ theories. We show that many of the phenomena in [3,4] are also present for generic $N = 1$ supersymmetric theories which have a Coulomb phase. In particular, we will argue that there are points on the moduli space with massless magnetic monopoles. With a suitable perturbation these monopoles condense and the theory passes to a confining phase.

In the third section we combine the techniques of [10] and those developed in [3,4] and in section 2 to analyze illustrative examples. The confining phase is well described by the superpotentials of [10]. These are not valid in the Coulomb phase or in a Higgs phase when it is distinct from the confining phase. The reason is that new degrees of freedom should be included for a proper description of the transition point. It has already been observed in [3,4], and we will show more generally in $N = 1$ examples below, that sometimes there is no Lagrangian which describes the low energy physics everywhere on the moduli space. We must be content with effective Lagrangians which describe only patches of the moduli space. In the overlap regions between different patches the different Lagrangians describe the same massless modes but include different massive modes.

2. The Coulomb phase

A gauge theory in the Coulomb phase has a massless photon and therefore it is subject to standard electric-magnetic duality. The gauge kinetic term in the low energy effective Lagrangian is

$$\frac{1}{64\pi} \text{Im} [\tau(g_I, \Phi^r)(F + i^*F)^2] = \frac{1}{32\pi} (\text{Im} \tau F^2 + \text{Re} \tau F^*F) \quad (2.1)$$

where τ gives the effective coupling $\tau = \frac{\theta_{eff}}{\pi} + i \frac{8\pi}{g_{eff}^2}$, g_I are the coupling constants of the underlying microscopic theory, and Φ^r are some light fields. (We are here normalizing τ as in [4] because we will consider examples with matter fields in the fundamental of $SU(2)$.) Under the electric-magnetic duality transformation S , the term (2.1) is mapped into

$$\frac{1}{64\pi} \text{Im} [\tau_d(F_d + i^*F_d)^2] \quad (2.2)$$

where F_d is the dual of F and $\tau_d = -1/\tau$. The effect on the spectrum is to take states with magnetic and electric charges (n_m, n_e) to $(n_e, -n_m)$. There is another duality transformation, denoted by T , which maps $\tau \rightarrow \tau + 1$ and has the effect of shifting θ . As in [12], this

changes the electric charges of states with magnetic charge as $T : (n_m, n_e) \rightarrow (n_m, n_e + n_m)$. Together, these two transformations generate the infinite duality group $SL(2, \mathbf{Z})$ (only $PSL(2, \mathbf{Z})$ acts on τ).

The function $\tau(g_I, \Phi^r)$ in (2.1) is not necessarily single valued. For example, as a microscopic θ parameter is shifted by 2π , $\text{Re } \tau$ is shifted by an integer. More generally, changing g_I and Φ^r along a closed path can transform τ and the spectrum by an $SL(2, \mathbf{Z})$ transformation. Therefore, τ is not a function but a section of an $SL(2, \mathbf{Z})$ bundle over the space of g_I and Φ^r . This was observed in [3,4] in an $N = 2$ supersymmetric context but is clearly more general. In non-supersymmetric theories it is not easy to determine $\tau(g_I, \Phi^r)$. However, as we will see below, $\tau(g_I, \Phi^r)$ can often be found exactly in $N = 1$ supersymmetric theories.

We consider $N = 1$ supersymmetric gauge theories based on gauge group \mathcal{G} (taken to be semi-simple) with matter fields ϕ_i in representations R_i of \mathcal{G} . Typically the potential has flat directions with nonzero ϕ_i expectation values. These break \mathcal{G} to a subgroup \mathcal{H} . This moduli space of classical ground states is labeled by the expectation values of various gauge invariant polynomials Φ^r of the ϕ_i . We will be interested in an effective low energy theory with the massive \mathcal{G}/\mathcal{H} vector bosons integrated out. If the remaining \mathcal{H} super Yang-Mills theory is non-abelian, it will dynamically generate a mass gap, confine, and the light fields in the low energy effective theory are simply the gauge singlets Φ^r . However, when \mathcal{H} contains an abelian factor, say a single $U(1)$, the spectrum includes a massless photon supermultiplet and the theory is in the Coulomb phase. The effective Lagrangian in this phase has a term which is the $N = 1$ supersymmetric version of (2.1)

$$\mathcal{L} = \dots + \frac{1}{16\pi} \text{Im} \int d^2\theta \tau(g_I, \Phi^r) W_\alpha W^\alpha, \quad (2.3)$$

where W_α is the photon field strength superfield.

Supersymmetry requires τ in (2.3) to be holomorphic¹ in the chiral superfields Φ^r and also in all the coupling constants g_I [2]. This holomorphy often enables us to determine τ exactly. This was done in [3,4] in some $N = 2$ theories and will be done here for some $N = 1$ theories. Although τ is a holomorphic function of its arguments, it is not single valued. Exactly as with (2.1), there is an electric-magnetic duality transformation which maps $W \rightarrow W_D$ and $\tau \rightarrow \tau_D = -1/\tau$; in addition there is the operation T of shifting the

¹ It is important here that we are discussing a Wilsonian effective action [13].

θ angle by 2π (or π). Once again, τ is a section of an $SL(2, \mathbf{Z})$ bundle over the space of Φ^r and g_I .

For simplicity, we will consider the case where there is a single τ which depends only on a single light field U . For large U the underlying \mathcal{G} gauge theory is weakly coupled and the one loop beta function in the microscopic theory leads to

$$\tau \approx \frac{iM}{2\pi} \log \frac{U}{\Lambda^b} \quad (2.4)$$

for some integers M and b . Λ is the scale of \mathcal{G} , which we assume here is simple (below we will also consider a semi-simple \mathcal{G} example). As we circle around infinity, $U \rightarrow e^{2\pi i} U$, $\tau \rightarrow \tau - M$; i.e. τ is transformed by $\mathcal{M}_\infty = T^{-M}$, which leaves the low energy effective gauge coupling $\frac{1}{g_{eff}^2} \sim \text{Im } \tau$ unchanged. The following argument shows that $\text{Im } \tau$ cannot be single valued in the interior of the moduli space [3,4]. If $\text{Im } \tau$ is single valued, it is a harmonic function, which cannot be positive definite. There would then be regions in the moduli space where g_{eff} is imaginary. This unphysical conclusion can be avoided if the topology of the moduli space is complicated in the interior or, as found in [3,4], there are several (at least two) singular values U_i of U with monodromies \mathcal{M}_i around them which do not commute with $\mathcal{M}_\infty = T^{-M}$.

The monodromies \mathcal{M}_i around the U_i must have a physical interpretation. The simplest one is that they are associated with k_i massless particles at the singularity. The low energy superpotential near U_i then has the form

$$W_L^{(i)} = (U - U_i) \sum_{l=1}^{k_i} c_l^{(i)} \tilde{E}_l^{(i)} E_l^{(i)} + \mathcal{O}((U - U_i)^2) \quad (2.5)$$

where $\tilde{E}_l^{(i)}$ and $E_l^{(i)}$ are the new massless states. If the constants $c_l^{(i)}$ are nonzero, these states acquire a mass of order $\mathcal{O}((U - U_i))$ away from the singularity. Therefore, the one loop beta function in the low energy theory leads to

$$\tau_i \approx -\frac{ik_i}{2\pi} \log(U - U_i) \quad (2.6)$$

(we assume for simplicity that, as in [3,4], all the $E_l^{(i)}$ have charge one; the generalization to other cases is straightforward) where τ_i is the coupling to the low energy photon. τ_i is related to τ in the asymptotic region by a duality transformation N_i . It is clear from (2.6) that the monodromy in τ_i is T^{k_i} . Therefore, the monodromy in τ is

$$\mathcal{M}_i = N_i^{-1} T^{k_i} N_i. \quad (2.7)$$

For \mathcal{M}_i to not commute with $\mathcal{M}_\infty = T^{-M}$, the transformation N_i must be non-trivial. This means that the massless particles $E_l^{(i)}$ at U_i are magnetically charged.

As discussed in [3,4], because τ is a section of an $SL(2, \mathbf{Z})$ bundle it is naturally interpreted as the modular parameter of a torus. A torus is conveniently described by the one complex dimensional curve in C^2 :

$$y^2 = x^3 + ax^2 + bx + c, \quad (2.8)$$

where $(x, y) \in C^2$ and a, b and c are parameters to be related to U and the various coupling constants and scales. The function τ is singular when the torus is singular, which is when

$$x^3 + ax^2 + bx + c = 0 \quad \text{and} \quad 3x^2 + 2ax + b = 0. \quad (2.9)$$

Eliminating x , this is when the discriminant of the cubic equation in (2.9) vanishes: $\Delta(a, b, c) = 0$ where

$$\Delta = 4a^3c - b^2a^2 - 18abc + 4b^3 + 27c^2. \quad (2.10)$$

As discussed in [3,4], the order of the zero can be used to determine the monodromy around the singularity.

Important constraints [4] on the dependence of the coefficients in (2.8) on U and the coupling constants are the following:

1. In the weak coupling limit $\Lambda = 0$ the curve should be singular for every U . Without loss of generality we can then take $y_0^2 = x^2(x - U)$.
2. The constants a, b, c in (2.8) are holomorphic in U and the various coupling constants. This guarantees that τ is holomorphic in them.
3. The curve (2.8) must be compatible with all the global symmetries of the theory including those which are explicitly broken by the coupling constants or the anomaly.
4. In various limits (e.g. as some mass goes to zero or infinity) we should recover the curves of other models.
5. The curve should have the correct monodromies around the singular points.

In the next section we will give several examples demonstrating how these constraints can be used to determine τ .

As in [3,4], we can add a superpotential W_{tree} to the microscopic theory and lift the flat directions. Adding it in the low energy theory to (2.5), we find that

$$\begin{aligned} \sum_{l=1}^{k_i} c_l^{(i)} \tilde{E}_l^{(i)} E_l^{(i)} + \frac{\partial W_{tree}}{\partial U} &= 0 \\ (U - U_i) E_l^{(i)} &= (U - U_i) \tilde{E}_l^{(i)} = 0 \\ \sum_l |E_l^{(i)}|^2 - |\tilde{E}_l^{(i)}|^2 &= 0. \end{aligned} \tag{2.11}$$

Therefore, for nonzero $\frac{\partial W_{tree}}{\partial U}$, at least one of the light charged field pairs $E_l^{(i)}$ and $\tilde{E}_l^{(i)}$ acquire expectation values, giving the photon a mass by the Higgs mechanism. Since the $E_l^{(i)}$'s are magnetic monopoles, their expectation values lead to confinement of the elementary fields. In addition, U is “locked” at the singularity U_i .

In many examples the expectation value of $U \frac{\partial W_{tree}}{\partial U}$ is closely related, by the Konishi anomaly [8], to the expectation value of the “glueball” superfield $S = -\frac{1}{32\pi^2} \text{Tr} W_\alpha^2$. $\langle S \rangle$ is then related by (2.11) to the expectation values of the monopoles $E_l^{(i)}$:

$$\langle S \rangle \sim \langle U \sum_{l=1}^{k_i} c_l^{(i)} \tilde{E}_l^{(i)} E_l^{(i)} \rangle. \tag{2.12}$$

Therefore, $\langle S \rangle = \langle \lambda \lambda \rangle / 32\pi^2 \neq 0$ only when monopoles condense; i.e. when the theory confines. There are however situations where the expectation value of $U \frac{\partial W_{tree}}{\partial U}$ is not simply related to $\langle S \rangle$ by the Konishi anomaly. This is the case in the last example in this paper; but there too we find that $\langle S \rangle$ is proportional to the monopole bilinear. It is therefore tempting to speculate that perhaps $\langle S \rangle$ can always be used as a local order parameter for confinement. There is also the possibility that $\langle U \frac{\partial W_{tree}}{\partial U} \rangle = \langle S \rangle = 0$ with nonzero monopole expectation values either because the terms in (2.11) sum to zero or because $\langle U \rangle = 0$. Both possibilities indeed occur in some of the theories of [4]. However, in those models there is no distinction between the Higgs phase and the confining phase and therefore we do not know if there is a counter example to this speculation.

The theory with $W_{tree} = 0$ has a Coulomb phase where there is a photon and the field U whose expectation value can be varied, serving as a moduli space coordinate for the Coulomb phase. When W_{tree} is turned on, the theory passes to the confining phase where U is locked at one of the singular values U_i , the photon is lifted, and charges are confined. The variable coordinates of the confining phase are the parameters in W_{tree} . That phase

has two equivalent descriptions. The first is the previous one involving the fields of the Coulomb phase and the light monopoles; this description has a smooth $W_{tree} \rightarrow 0$ limit. The other is in terms of the gauge invariant fields of the confining phase; it is not valid at the transition point.

Techniques for finding a gauge invariant description of the confining phase were discussed in [10]. For the general situation where there can be a Coulomb phase and the confining phase and the Higgs phase can be distinct, these techniques only probe the confining phase and thus lead to incomplete results. For example, they are bound to give a constraint which fixes U to be at one of the singular values U_i , which is correct for the confining phase but misses the U moduli space of the Coulomb phase. In the examples in the next section we will show that this description of the confining phase is connected properly to the Coulomb phase.

To summarize, it is generic to find points on the moduli space with massless magnetic monopoles. Furthermore, appropriate perturbations in the microscopic theory lead to condensation of these monopoles and confinement. These observations in [3,4] are thus more general and are not limited to theories with $N = 2$ supersymmetry. However, unlike the $N = 2$ theories, our discussion only concerns τ and does not determine the dyon masses, nor does it determine the Kahler potential and the metric on the moduli space.

3. Examples

3.1. $SU(2)$ with a triplet ϕ

This is the example considered in [3]. The gauge singlet superfield $U = \text{Tr}\phi^2$ is a coordinate on the quantum moduli space. As discussed in [4], the curve (2.8) which gives τ in the Coulomb phase is

$$y^2 = x^3 - x^2U + \frac{1}{4}\Lambda^4x. \quad (3.1)$$

Turning on a superpotential $W_{tree} = mU$ gives the field ϕ a mass. Below the scale m , ϕ can be integrated out and the theory becomes $N = 1$, $SU(2)$ Yang-Mills theory. The Konishi anomaly [8] gives $\langle \phi \frac{\partial W_{tree}}{\partial \phi} \rangle = 4\langle S \rangle$ where the factor of 4 arises from the index of the adjoint representation. This gives $m\langle U \rangle = \pm 2\Lambda_d^3$ where Λ_d is the scale of the low energy $SU(2)$ theory and we used the known $SU(2)$ gaugino condensation result $\langle S \rangle = \pm \Lambda_d^3$. This constraint on the value of U is indeed appropriate for the confining branch: these are the values $U_i = \pm \Lambda^2$, of U where (3.1) is singular. The matching relation between the scale

Λ_d and the scale Λ of the high-energy theory which includes ϕ is thus $\Lambda_d^6 = \frac{1}{4}m^2\Lambda^4$, where the 4 reflects the above normalization conventions, chosen to agree with those of [4,10].

The constraint on U on the confining branch can be nicely seen by starting from the superpotential for the massive field S of $SU(2)$ Yang-Mills theory [5]

$$W_d = S \left(\log \left(\frac{m^2 \Lambda^4}{4S^2} \right) + 2 \right), \quad (3.2)$$

where we used the above relation between Λ_d and Λ . As discussed in [10], the superpotential $W = W_u + mU$ of the theory which includes ϕ can be determined by “integrating in” ϕ , using $W_u = [W_d - mU]_{\langle m \rangle}$ where $\langle m \rangle$ solves $\frac{\partial}{\partial m}(W_d - mU) = 0$; this is the inverse of integrating out ϕ : $W_d = [W_u + mU]_{\langle U \rangle}$. The result for $W = W_u + mU$ is then

$$W = S \log \left(\frac{\Lambda^4}{U^2} \right) + mU. \quad (3.3)$$

The equations of motion give

$$\begin{aligned} U &= \pm \Lambda^2 \\ S &= \frac{1}{2}mU. \end{aligned} \quad (3.4)$$

Note that, as in the discussion after (2.11), $\langle S \rangle$ is an order parameter for confinement in this theory.

These results are indeed correct for $m \neq 0$. On the other hand, the $m \rightarrow 0$ limit of (3.3) continues to give the constraint $U = \pm \Lambda^2$ of the confining branch, whereas U is unconstrained on the Coulomb branch. We conclude that (3.3) is valid only on the confining branch. It has already been observed in [3,4] that often effective Lagrangians are valid only in patches and not everywhere on the space of parameters and moduli.

3.2. $SU(2)_1 \times SU(2)_2$ with two fields Q_f ($f = 1, 2$) in the $(\mathbf{2}, \mathbf{2})$

The gauge singlets are $M_{fg} = Q_f \cdot Q_g \equiv \frac{1}{2}Q_{f,c_1c_2}Q_{g,d_1d_2}\epsilon^{c_1d_1}\epsilon^{c_2d_2}$ in the $\mathbf{3}$ of the global $SU(2)_F$ flavor symmetry. The theory has a 3 complex dimensional moduli space of vacua labeled by M_{fg} ; these vacua remain exactly degenerate in the quantum theory because the symmetries do not allow any superpotential to be generated. Classically, when $U \equiv \det M_{fg} \neq 0$ the $SU(2)_1 \times SU(2)_2$ gauge group is broken by the Higgs mechanism down to a $U(1)_D \subset SU(2)_D$, where $SU(2)_D$ is a diagonally embedded $SU(2)$ subgroup. Therefore, the theory is in the Coulomb phase.

By the $SU(2)_F$ flavor symmetry, the function τ of the Coulomb phase is $\tau(U, \Lambda_1^4, \Lambda_2^4)$, where the scales Λ_1 and Λ_2 refer to $SU(2)_1$ and $SU(2)_2$ and the exponents are those of instanton contributions, $e^{-8\pi^2/g_i^2(\mu)} = (\Lambda_i/\mu)^4$. Consider the limit of large U . Taking M_{11} large, the gauge symmetry is broken to $SU(2)_D$, with Q_2 decomposing into a singlet and a triplet ϕ_D , and there is also the singlet M_{11} . The low energy theory is then similar to the previous example with the light field $U_D = \text{Tr}\phi_D^2 = 2U/M_{11}$, where the 2 is from the trace, and with a scale Λ_D related to the Λ_i by² $\Lambda_D^4 = 16\Lambda_1^4\Lambda_2^4/M_{11}^2$. In this limit, τ is therefore given by the curve (3.1) for U_D and Λ_D ; this gives

$$y^2 = x^3 - x^2U + x\Lambda_1^4\Lambda_2^4 \quad \text{for large } U, \quad (3.5)$$

where we used the above relations and we rescaled $x \rightarrow 2x/M_{11}$ and $y \rightarrow (2/M_{11})^{3/2}y$. This curve gives

$$\tau \approx \frac{i}{\pi} \log \left(\frac{U^2}{\Lambda_1^4\Lambda_2^4} \right) \quad \text{for large } U; \quad (3.6)$$

so taking $U \rightarrow e^{2\pi i}U$ gives a monodromy of $\mathcal{M}_\infty = T^{-4}$. This monodromy means that there must be (at least two) strong coupling singular points where monopoles become massless.

Using the symmetries, including the \mathbf{Z}_2 symmetry which exchanges the two gauge groups, the exact coefficient a in (2.8) must be $a = -U + \alpha(\Lambda_1^4 + \Lambda_2^4)$, for some constant α . In addition, agreement with (3.5) requires $b = \Lambda_1^4\Lambda_2^4$ and $c = 0$ to be exact expressions. Note that this curve is always singular when either Λ_1 or Λ_2 vanishes. This reflects the fact that the low energy photon decouples, and hence $\tau \rightarrow i\infty$, in these limits. To determine the remaining parameter α , we consider the limit $\Lambda_2 \gg \Lambda_1$. In this limit the theory is approximately an $SU(2)_1$ gauge theory with three singlets M_{fg} as well as a field $\tilde{\phi}$ in the adjoint of $SU(2)_1$. These fields are related by the constraint of [1]

$$\text{Pf } V = U + \mu^2\tilde{U} = \Lambda_2^4, \quad (3.7)$$

where μ is a dimensionful normalization needed to make $\tilde{\phi}$ a canonical field. $SU(2)_1$ with a field $\tilde{\phi}$ in the adjoint is singular at $\tilde{U} = \pm\Lambda_1^2$. Using (3.7), τ should then be singular at $U \approx \Lambda_2^4 \pm \mu^2\Lambda_1^2$ in the $\Lambda_2 \gg \Lambda_1$ limit. On the other hand, the discriminant reveals that

² The threshold factor 16 can be determined by giving Q_2 a mass m . In the resulting low energy theory [10], $\tilde{\Lambda}_1^5\tilde{\Lambda}_2^5/M_{11}^2 = \tilde{\Lambda}_D^6$. As usual for matter in the fundamental, $\tilde{\Lambda}_i^5 = m\Lambda_i^4$. But our matching condition for adjoint fields gives $\tilde{\Lambda}_D^6 = \tilde{m}^2\Lambda_D^4/4$ where $\tilde{m} = mM_{22}/U_D = m/2$.

our curve is singular at $U = \alpha(\Lambda_1^4 + \Lambda_2^4) \pm 2\Lambda_1^2\Lambda_2^2$. Comparing, we find that $\alpha = 1$. To summarize, we have determined that τ is given by the curve

$$y^2 = x^3 + x^2(-U + \Lambda_1^4 + \Lambda_2^4) + \Lambda_1^4\Lambda_2^4x. \quad (3.8)$$

In addition to the weak coupling singularity (3.6), this τ is singular for two values of $\det M \equiv U$ in the strong coupling region: $U_i = (\Lambda_1^2 \pm \Lambda_2^2)^2$. Note that in terms of the moduli space of vacua given by the expectation values of the M_{fg} , these are singular (non-compact) submanifolds rather than singular points. The order of the zero of the discriminant at these singularities implies that they both have monodromy conjugate to T . There is thus a single massless field on each of the two U_i submanifolds. Their charges are $(n_m, n_e) = (1, 0)$ for one of the singular spaces and $(1, 1)$ for the other, much as in [3].

Near either of the two strong coupling singular submanifolds of $\det M = U_i = (\Lambda_1^2 \pm \Lambda_2^2)^2$, the low energy theory is approximately described by the effective superpotential

$$W^{(i)} \approx c^{(i)}(\det M - U_i)\tilde{E}^{(i)}E^{(i)} + \text{Tr } mM. \quad (3.9)$$

For $m \neq 0$, the equations of motion give $M = \epsilon\sqrt{\det m U_i}m^{-1}$ and $c^{(i)}\tilde{E}^{(i)}E^{(i)} = -\epsilon\sqrt{\frac{\det m}{U_i}}$, where $\epsilon = \pm 1$. So each singular submanifold U_i gives two vacua in the fully massive theory, for a total of four vacua. In the limit of large m the Q_f can be integrated out and these four vacua go over to the four vacua of the low energy $SU(2) \times SU(2)$ Yang-Mills theory. As in the previous example, the condensation of the monopoles leads to confinement. Even though the theory has matter fields in the fundamental representation of each $SU(2)$ factor, the Higgs and confining phases are different. All the matter fields are invariant under the diagonal \mathbf{Z}_2 subgroup of the centers of the two $SU(2)$ factors. Therefore, Wilson loops in representations which are affected by this \mathbf{Z}_2 , say $(\mathbf{2}, \mathbf{1})$, should exhibit area law in the confining phase.

The singular values U_i of U can also be determined directly by giving the Q_f masses and considering the confining phase, where U is automatically locked at the U_i . Starting from the low-energy $SU(2)_1 \times SU(2)_2$ Yang-Mills theory, the Q_f can be integrated in using the technique of [10]. This is essentially the reverse of the discussion of the previous paragraph. The result is that the confining branch is described by

$$W = S_1 \log \left(\frac{\Lambda_1^4(S_1 + S_2)^2}{S_1^2 \det M} \right) + S_2 \log \left(\frac{\Lambda_2^4(S_1 + S_2)^2}{S_2^2 \det M} \right) + \text{Tr } mM. \quad (3.10)$$

Upon integrating out the S_s , this gives

$$W = \text{Tr } mM \quad \text{with} \quad \det M = U_{\pm} \equiv (\Lambda_1^2 \pm \Lambda_2^2)^2. \quad (3.11)$$

For example, giving a mass m_2 to the matter field Q_2 and integrating it out subject to (3.11) gives the effective Lagrangian found in [10] for the remaining light field M_{11} . The vacua have $\langle S_s \rangle = \epsilon_s \sqrt{\det m \Lambda_s^4}$, with $\epsilon_1 \epsilon_2 = \pm 1$ for $\langle U \rangle = (\Lambda_1^2 \pm \Lambda_2^2)^2$. Again we see that these are order parameters for the confinement which occurs for nonzero $\det m$.

In the confining phase (3.11), as discussed in the previous section, U is indeed locked at the singular values U_i of the Coulomb phase determined from the curve (3.8). In fact, having determined the U_i by thus analyzing the confining phase, we could have bypassed some of the previous detailed analysis of the curve by knowing the values of the U_i .

3.3. $SU(2)$ with two triplets ϕ_f ($f = 1, 2$)

The classical moduli space is parametrized by the expectation values of the gauge singlet fields $M_{fg} = \text{Tr}(\phi_f \phi_g)$, which transform in the **3** of the global $SU(2)_F$ flavor symmetry. For generic values of M the $SU(2)$ gauge symmetry is completely broken and the theory is in the Higgs phase. On the submanifold with $\det M = 0$ there is an unbroken $U(1)$ gauge symmetry and thus a light photon – this subspace is in the Coulomb phase.

This vacuum degeneracy cannot be lifted quantum mechanically. The reason is that the only invariant superpotential for the light fields is proportional to $\det M/\Lambda$. This does not have a proper semi-classical limit ($\Lambda \rightarrow 0$) and therefore cannot be generated to lift the degeneracy of the Higgs phase or the Coulomb phase subspace. Below we will argue that such a superpotential does give a proper description of the confining phase.

Now consider adding a tree level superpotential $W_{tree} = \text{Tr } mM$. For $\det m \neq 0$ it gives both ϕ_f a mass and the low energy theory is an $N = 1$ pure gauge $SU(2)$ theory, which is known to confine. For $m \neq 0$ but $\det m = 0$ only one of the matter fields gets a mass and the low energy theory is an $N = 2$ pure gauge theory, which is in the Coulomb phase [3]. We see that the theory can be in all three of the different possible phases. Note that, because the theory contains no fields in the fundamental representation of the gauge group, the confining and Higgs phases are distinct [11] – the Wilson loop has an area law in the confining phase and a perimeter law in the Higgs phase.

The confining phase, using the technique of [10], is described by

$$W = S \left[\log \left(\frac{\Lambda^2 S^2}{\det M^2} \right) - 2 \right] + \text{Tr } mM. \quad (3.12)$$

Here we used the matching relation $\Lambda_d^6 = \det m^2 \Lambda^2 / 16$ between the scale Λ of the high-energy theory which includes the two ϕ_i and the scale Λ_d of the low-energy $N = 1$ pure gauge $SU(2)$ theory (this relation is the one discussed for $SU(2)$ with a single adjoint, applied for each eigenvalue of m). Integrating out the field S by its equation of motion gives $\langle S \rangle = \pm \Lambda^{-1} \langle \det M \rangle$ and the superpotential

$$W = \mp 2 \frac{\det M}{\Lambda} + \text{Tr } mM. \quad (3.13)$$

Upon integrating out M , we obtain

$$\langle M \rangle = \pm \frac{1}{2} (\Lambda \det m) m^{-1} \quad \text{and} \quad \langle S \rangle = \pm \frac{1}{4} \Lambda \det m. \quad (3.14)$$

These are the correct expectation values for the confining phase.

However if we take $m = \begin{pmatrix} 0 & 0 \\ 0 & m_{22} \end{pmatrix}$, giving mass to ϕ_2 only, the theory is actually in the Coulomb phase. Nevertheless, (3.14) gives $\text{Tr} \phi_1^2 = \pm \frac{1}{2} m_{22} \Lambda = \pm \Lambda_I^2$ where Λ_I , the scale of the theory with ϕ_2 integrated out, is related to Λ by the matching relation for adjoint fields. We see, once again, that continuing from the confining phase gives only the singular values U_i , missing the Coulomb phase moduli space of U . In this phase, as in (3.1), the curve which determines τ is

$$y^2 = x^2(x - M_{11}) + \frac{1}{16} m_{22}^2 \Lambda^2 x. \quad (3.15)$$

It is singular as $m_{22} \rightarrow 0$. This is because there are massless charged “electrons” on the Coulomb branch for $m = 0$. The low energy gauge coupling is therefore renormalized to zero in the infra-red. Hence $\tau \rightarrow i\infty$ and the curve must be singular.

We can now see the transitions between the various phases. For $m = 0$ the $\det M = 0$ submanifold of the moduli space is in the Coulomb phase and there are charged “electrons” on this submanifold. The transition to the Higgs phase is characterized by the expectation values of these electrons. For $m \neq 0$ but $\det m = 0$ the theory has only a Coulomb phase while for $\det m \neq 0$ there is only a confining phase. The transition between them takes place by monopole condensation at $M = \pm \frac{1}{2} (\Lambda \det m) m^{-1}$. Again, the expectation value (3.14) of S is a local order parameter for the $\det m \neq 0$ confining phase, as expected from the discussion following (2.11).

A direct transition from the Higgs phase to the confining phase happens at $M = 0$ for $m = 0$. The simplest interpretation of the physics at that point is that the elementary gauge bosons of the underlying $SU(2)$ theory are massless there. Hence, this theory must be at a non-trivial fixed point of the renormalization group. Examples of such fixed points have already appeared in [1].

3.4. $SU(2)$ with two doublets Q_f ($f = 1, 2$) and a triplet ϕ

The basic gauge singlets are $X = Q_1 Q_2$, $U = \text{Tr} \phi^2$, and $\vec{Z} = \frac{\sqrt{2}}{2} Q_f \phi Q_g \vec{\sigma}^{fg}$ transforming in the **3** of the global $SU(2)_F$ flavor symmetry. Classically, there is a moduli space of inequivalent vacua given by the expectation values of these fields subject to the classical constraint $\vec{Z}^2 = X^2 U$. In the quantum theory the above five fields are independent with a superpotential which is determined by the symmetries to be of the form

$$W_u = -\frac{XU^2}{\Lambda^3} f\left(t = \frac{\vec{Z}^2}{X^2 U}\right). \quad (3.16)$$

We will determine the function $f(t)$ shortly. In order to reproduce the correct asymptotic behavior of the moduli space, the function $f(t)$ in (3.16) must have a double zero at $t = 1$; $f(1) = f'(1) = 0$. There is then a quantum moduli space of vacua characterized by $t = 1$; i.e. by $\vec{Z}^2 = X^2 U$. For generic expectation values of the fields the $SU(2)$ gauge symmetry is completely broken and the theory is in the Higgs phase. Note that, since there is no distinction between the Higgs and the confining phase for this theory, we expect (3.16) to describe the theory everywhere away from the Coulomb phase.

The quantum moduli space is singular on the subspace $X = \vec{Z} = 0$ for any U . On that complex line there is an unbroken $U(1)$ gauge symmetry and the theory is in the Coulomb phase. The physical reason for the singularity is that X and the \vec{Z} are not the correct light fields on this submanifold: in the Coulomb phase the elementary charged fields Q_f are massless. As in the previous example, these fields get expectation values off the Coulomb submanifold, leading to a transition to the Higgs phase.

Consider turning on a tree level superpotential $W_{tree} = m_Q X + \vec{\lambda} \cdot \vec{Z}$. As long as either m_Q or $\vec{\lambda}$ is nonzero, this superpotential fixes the theory to lie on the Coulomb phase submanifold. For $\vec{\lambda}^2 = 1$, the theory is $N = 2$ supersymmetric and was analyzed in [4]. It was found there that the function τ is described by the curve

$$y^2 = x^2(x - U) + \frac{1}{4}m_Q\Lambda^3x - \frac{1}{64}\Lambda^6 \quad \text{for} \quad \vec{\lambda}^2 = 1. \quad (3.17)$$

This can be immediately generalized to arbitrary $\vec{\lambda}^2$ (it only depends on $\vec{\lambda}^2$ by the $SU(2)_F$ flavor symmetry). The theory has a global $U(1)_Q \times U(1)_\phi \times U(1)_R$ symmetry with the charges $U : (0, 2, 0)$, $m_Q : (-2, 0, 2)$, $\vec{\lambda}^2 : (-4, -2, 4)$, and $\Lambda^3 : (2, 4, -2)$ (using the anomaly as in [10]). The terms in (3.17) should thus have charges $(0, 6, 0)$ and hence

$$y^2 = x^3 - x^2 U + \frac{1}{4}m_Q\Lambda^3x - \frac{1}{64}\vec{\lambda}^2\Lambda^6. \quad (3.18)$$

This curve is singular when $m_Q = \vec{\lambda} = 0$. In this case there are massless “electrons” in the Coulomb phase and they renormalize the electric charge to zero in the infra-red. Hence $\tau = i\infty$ and the curve is singular.

In the Higgs phase, adding $W_{tree} = m_Q X + \vec{\lambda} \cdot \vec{Z}$, the theory is described by the superpotential $W = -\frac{XU^2}{\Lambda^3} f(t) + W_{tree}$. On the other hand, because there are matter fields in the fundamental representation of the gauge group, there is no phase boundary separating the Higgs phase and the confining phase. Therefore, we can approach the Higgs phase from the Coulomb line via the confining phase. The expectation value of the field U should then be at the singular values U_i obtained from the curve (3.18). Therefore, the equations of motion obtained from this Higgs phase superpotential must fix U at the singular values U_i of the curve (3.18). Indeed, upon integrating out the massive fields X and \vec{Z} , the Higgs phase superpotential becomes $W = 0$ with the constraints

$$U^2(f - 2tf') = m_Q \Lambda^3 \equiv 4b \quad \text{and} \quad 4U^3 f'^2 t = \vec{\lambda}^2 \Lambda^6 \equiv -64c. \quad (3.19)$$

Eliminating t , these equations fix U to particular values U_i . On the other hand, using (3.18) and (2.10), the singular values U_i of U satisfy

$$-4U^3 c - U^2 b^2 + 18Ubc + 4b^3 + 27c^2 = 0. \quad (3.20)$$

The equations (3.19) and (3.20) must agree for every value of the parameters b and c . This gives a differential equation for f :

$$4f'^2 t - (f - 2tf')^2 - \frac{9}{2}(f - 2tf')f'^2 t + (f - 2tf')^3 + \frac{27}{16}f'^4 t^2 = 0. \quad (3.21)$$

This differential equation has a unique solution subject to the boundary conditions $f(1) = f'(1) = 0$ discussed above. The unique solution is

$$f(t) = (1 - t)^2. \quad (3.22)$$

The equations (3.19) with $f = (1 - t)^2$ are

$$U^2(1 - t)(1 + 3t) = m_Q \Lambda^3 \quad \text{and} \quad 16U^3 t(1 - t)^2 = \vec{\lambda}^2 \Lambda^6; \quad (3.23)$$

they are equivalent to the singularity equations (2.9) with the substitution $2x = U(1 - t)$. Note that, for m_Q or $\vec{\lambda}$ nonzero, $t \neq 1$; adding m_Q or $\vec{\lambda}$ takes the theory off its quantum moduli space. This phenomenon has already been observed in [1,10,4]. For example, for

$\vec{\lambda} = 0$ with $m_Q \neq 0$, (3.23) gives $U_{1,2} = \pm \sqrt{m_Q \Lambda^3}$ with $t = 0$ and $U_3 = \infty$ with $t = 1$ (as expected classically since the coupling is weak at $U = \infty$). For $m_Q = 0$ and $\vec{\lambda} \neq 0$, (3.23) gives $U_{1,2,3}^3 = -\frac{27}{256} \vec{\lambda}^2 \Lambda^6$ with $t = -\frac{1}{3}$. The monodromy around each of the singular values U_i of U is conjugate to T and, thus, a single light field is present at each U_i .

The superpotential (3.16) with $f(t) = (1-t)^2$ can also be obtained by considering the confining phase. Consider turning on the tree-level superpotential $W_{tree} = m_\phi U + \vec{\lambda} \cdot \vec{Z}$ and integrating out ϕ . Doing so at tree level, we obtain $W_{tree,d} = -\frac{\vec{\lambda}^2}{4m_\phi} X^2$. Including the gauge dynamics, the full superpotential in the theory with ϕ integrated out is uniquely determined, as in [10], to be simply $[W_u + m_\phi U + \vec{\lambda} \cdot \vec{Z}]_{\langle U \rangle, \langle \vec{Z} \rangle} = \frac{m_\phi^2 \Lambda^3}{4X} - \frac{\vec{\lambda}^2}{4m_\phi} X^2$. Any modification of this result can be ruled out using the symmetries, holomorphy, and the behavior in various limits. Thus $W_u = [\frac{m_\phi^2 \Lambda^3}{4X} - \frac{\vec{\lambda}^2}{4m_\phi} X^2 - m_\phi U - \vec{\lambda} \cdot \vec{Z}]_{\langle m_\phi \rangle, \langle \vec{\lambda} \rangle}$ and hence

$$W_u = -\frac{XU^2}{\Lambda^3} \left(1 - \frac{\vec{Z}^2}{X^2 U} \right)^2, \quad (3.24)$$

reproducing the result $f = (1-t)^2$. Having obtained this result, we can add $W_{tree} = m_Q X + \vec{\lambda} \cdot \vec{Z}$, use the equations of motion (3.23) to find the singular values U_i of U , and thus reproduce the answer obtained from the curve (3.18) of the Coulomb branch of the theory. Note that it is possible to explore the entire Higgs phase moduli space of $t = 1$ from the confining phase expectation values obtained by perturbing the theory by $W_{tree} = m_\phi U + m_Q X + \vec{\lambda} \cdot \vec{Z}$. In particular, taking $m_Q, m_\phi, \vec{\lambda} \rightarrow 0$ with $\vec{\alpha} = \vec{\lambda}/m_Q, \tilde{m}_\phi = m_\phi/m_Q$ fixed, we find $\langle 1-t \rangle \approx \Lambda^3 m_Q (\vec{\alpha}^2)^2/4 \rightarrow 0$, $\langle U \rangle \approx 1/\vec{\alpha}^2$, $\langle X \rangle \approx 2\tilde{m}_\phi/\vec{\alpha}^2$, and $\langle \vec{Z} \rangle \approx -2\vec{\alpha}\tilde{m}_\phi/(\vec{\alpha}^2)^2$. By adjusting $\vec{\alpha}$ and \tilde{m}_ϕ in this limit, these expectation values explore the entire $t = 1$ moduli space.

The superpotential with the massive field S integrated in is

$$W_u = S \left[\log \left(\frac{\Lambda^3 S}{XU^2(1-t)^2} \right) - 1 \right]; \quad (3.25)$$

this gives (3.24) upon integrating out S by its equation of motion $\langle S \rangle = XU^2(1-t)^2 \Lambda^{-3}$. Perturbing by $W_{tree} = m_\phi U + m_Q X + \vec{\lambda} \cdot \vec{Z}$ and integrating out X and \vec{Z} gives

$$W = S \log \left(\frac{m_Q \Lambda^3}{U^2(1-t_0)(1+3t_0)} \right) + m_\phi U \quad \text{with} \quad t_0^{-1}(1+3t_0)^2 = \frac{16m_Q^2}{\vec{\lambda}^2 U}, \quad (3.26)$$

which is equivalent to (3.23) upon integrating out S . Integrating out U gives $\langle S \rangle$ which is proportional to m_ϕ showing, again, that it is an order parameter for confinement in this theory.

There is one point on the moduli space which we have not yet discussed. For $m_Q = m_\phi = \vec{\lambda} = 0$ there is a ground state with $X = U = \vec{Z} = 0$. This point can be approached from several directions. Moving in from nonzero U with $X = \vec{Z} = 0$ we conclude that there must be a photon and massless charged fields there. If we instead set $m_\phi = 0$ and take m_Q and $\vec{\lambda}$ to zero in various ratios we find at $U = 0$ a photon and either two or three monopoles. We expect that the resolution is that the correct degrees of freedom at that point are actually the elementary quarks and gluons and that this is another scale invariant theory.

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